

# Quantum Dynamics of Mesoscopic Driven Duffing Oscillators

Lingzhen Guo, Zhigang Zheng, and Xin-Qi Li\*

Department of Physics, Beijing Normal University, Beijing 100875, China

(Dated: May 25, 2010)

We investigate the nonlinear dynamics of a mesoscopic driven Duffing oscillator in a quantum regime. In terms of Wigner function, we identify the nature of state near the bifurcation point, and analyze the transient process which reveals two distinct stages of quenching and escape. The rate process in the escape stage allows us to extract the transition rate, which displays perfect scaling behavior with the driving distance to the bifurcation point. We numerically determine the scaling exponent, compare it with existing result, and propose open questions to be resolved.

PACS numbers: 05.45.-a, 03.65.Xp, 85.25.Cp

A broad class of physical systems such as Josephson junction, trapped electron or ion, and nano-mechanical oscillator, can be well described by the Duffing oscillator under proper conditions. One of the most profound features of a driven Duffing oscillator (DDO) is the dynamical bifurcation. Near the bifurcation point, the oscillator state is highly sensitive to perturbation. This property can be exploited for applications such as sensing device, amplifier, and logic device. Most recently, for instance, the superconductor circuit based on Josephson junction has been exploited for quantum measurement of superconducting qubits [1–4]. This device is termed as Josephson bifurcation amplifier (JBA), holding advantages such as fast speed, high sensitivity, low backaction, and absence of on-chip dissipation.

Despite that the classical bifurcation of DDO is well-known, the quantum dynamics in the bistable region and near the bifurcation point has been a new and significant subject in the past years [5–10]. This new trend is motivated mostly by the advent of approaching the quantum regime of nano-mechanical oscillators, as well as the bifurcation-based quantum measurement devices. For instance, the quantum signature in the bistable region of a DDO was proposed, based on simulating a Lindblad-type master equation and comparing the Wigner function with classical probability distribution in phase space [5]. In terms of amplitude and phase responses to the driving frequency, quantum behaviors of DDO such as resonant tunneling and photon-assisted tunneling were also discussed [6]. Moreover, in Ref. [7–9], switching rate between the bistable states near the bifurcation point, due to quantum and/or thermal fluctuations, is estimated by means of the WKB theory or semiclassical methods such as the mean-first-passage-time approach.

In this letter we consider a *mesoscopic* DDO, with about more than ten levels that is in between the quantum few-level and the classical dense-level (or continuum) limits. In this regime, the quantum effect is apparently significant. However, at the same time, how the DDO's nonlinearity manifests itself is unclear and of interest, since the few-level (e.g. 2- or

3-level) system should have no such behaviors as bistability and bifurcation. Our present study will demonstrate the existence of bistable region, characterize the quantum nature of the states, and investigate the quantum transition near the classical bifurcation point which displays *perfect* and *new* scaling behavior with the driving strength.

*Model and Method.*— The Duffing oscillator in the presence of driving is described by the Hamiltonian

$$\hat{H}_0(t) = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\Omega^2\hat{x}^2 - \gamma\hat{x}^4 + F(t)\hat{x}. \quad (1)$$

For the JBA setup,  $F(t) = F_0(e^{i\omega t} + e^{-i\omega t})$  describes the microwave driving. Other parameters are related to the JBA circuit quantities as:  $m = (\hbar/2e)^2 C$ ,  $\Omega = \sqrt{2eI_c/(\hbar C)}$ ,  $F_0 = \hbar I/(2e)$ , and  $\gamma = m\Omega^2/24$ ; with  $C$  the capacitance of the Josephson junction,  $I_c$  the critical current, and  $I$  the driving current. In this context,  $x$  denotes the phase difference across the Josephson junction.

In addition, the Duffing oscillator is affected by environment, which together with the coupling can be modelled as  $\hat{H}_E = \sum_i [m_i\omega_i^2\hat{x}_i^2/2 + \hat{p}_i^2/2m_i] - \hat{x} \sum_i \lambda_i \hat{x}_i + \hat{x}^2 \sum_i \lambda_i^2/(2m_i\omega_i^2)$ . Typically, the spectral density of the bath,  $J(\omega) = \pi \sum_i \lambda_i^2 \delta(\omega - \omega_i)/(2m_i\omega_i)$ , in Ohmic case reads  $J(\omega) = m\kappa\omega \exp(-\omega/\omega_c)$ , with  $\kappa$  the friction coefficient, and  $\omega_c$  the high frequency cut-off. For later use, we also introduce  $\hat{b} = \sum_i \lambda_i \hat{b}_i/\sqrt{2}$ , with  $\hat{b}_i = (m_i\omega_i\hat{x}_i + i\hat{p}_i)/\sqrt{2m_i\hbar\omega_i}$ .

In the weak coupling limit to the environment and under Markovian approximation, the dissipative dynamics of the DDO is governed by the quantum master equation (see Ref. [11] for more details)

$$\dot{\rho}(t) = -\frac{i}{\hbar}[\hat{H}(t), \rho(t)] - \frac{1}{\hbar^2}\{[\hat{x}, \hat{Q}\rho(t)] + \text{H.c.}\}. \quad (2)$$

Here,  $\rho(t)$  is the reduced density matrix of the oscillator;  $\hat{H}(t) = \hat{H}_0(t) + \hat{x}^2 m\kappa\omega_c/\pi$ , and  $\hat{Q} = [C(-\mathcal{L}) + \tilde{C}(-\mathcal{L})]\hat{x}/2$ . The Liouvillian  $\mathcal{L}$  is defined through its action on an arbitrary operator  $\hat{O}$  as:  $\mathcal{L}\hat{O} \equiv \hbar^{-1}[\hat{H}(t) - F(t)\hat{x}, \hat{O}]$ . The superoperators  $C(\mathcal{L})$  and  $\tilde{C}(\mathcal{L})$  are the Fourier transform of the bath correlation functions:  $C(\mathcal{L}) = \int_{-\infty}^{+\infty} dt C(t)e^{i\mathcal{L}t}$ , and  $\tilde{C}(\mathcal{L}) = \int_{-\infty}^{+\infty} dt \tilde{C}(t)e^{i\mathcal{L}t}$ . The correlators  $C(t)$  and  $\tilde{C}(t)$  are defined by  $C(t) = \text{Tr}_E[\hat{b}^\dagger(t)\hat{b}(0)\rho_E]$ , and  $\tilde{C}(t) = \text{Tr}_E[\hat{b}(t)\hat{b}^\dagger(0)\rho_E]$ , where

\*Corresponding author: xqli@red.semi.ac.cn

$\rho_E$  is the thermal-equilibrium density operator of the environment.

*Qualitative Considerations.*— In the absence of driving, the Duffing oscillator described by Eq. (1) has only finite number of bound states. This can be seen from the potential profile,  $V(x) = m\Omega^2 x^2/2 - \gamma x^4$ , which defines a single well with identical barrier height  $V_0 = m^2\Omega^4/(16\gamma)$  at  $x = \pm\sqrt{m\Omega^2/(4\gamma)}$ . As a rough estimate, the number of bound states is the ratio of  $V_0$  and  $\hbar\Omega$ , which gives  $N = m^2\Omega^4/(16\gamma\hbar\Omega) = m\Omega/(16\hbar\tilde{\gamma}) = \aleph/(16\tilde{\gamma})$ . We will see later that  $\aleph \equiv m\Omega/\hbar$  defined here is an important characteristic quantity. We also introduced a reduced nonlinear coefficient,  $\tilde{\gamma} = \gamma/(m\Omega^2)$ . In our model,  $\gamma = m\Omega^2/24$ , so approximately the number of bound state is  $3\aleph/2$ . In the experiment of Ref. [1],  $\aleph \approx 366$ , which implies a classical DDO. In the present work, we consider a *mesoscopic* regime, by assuming possible parameters  $I_c = 39\text{nA}$ ,  $C = 0.91\text{pF}$ ,  $\kappa = 0.01\Omega$ , and  $\omega_c = 10\Omega$ . Accordingly,  $\aleph \approx 12$ .

Under proper conditions [3], the DDO exhibits the most profound phenomenon known as *bifurcation*. To determine the bifurcation point, we present an analysis in the *rotating phase space*. Starting with the Hamiltonian  $H_0(t)$ , we introduce a unitary transformation,  $\hat{U} = \exp[ivt(\hat{a}^\dagger\hat{a})]$ , where  $\hat{a}$  is the annihilation operator of the Duffing oscillator. Under the rotating wave approximation (RWA), we obtain [9]

$$\hat{H}_s^{(\delta)} = \left( \frac{\hat{p}^2}{2\tilde{m}} + \frac{1}{2}\tilde{m}\tilde{\Omega}^2\hat{x}^2 \right) - \frac{6\gamma}{4\tilde{m}^2\tilde{\Omega}^4} \left( \frac{\hat{p}^2}{2\tilde{m}} + \frac{1}{2}\tilde{m}\tilde{\Omega}^2\hat{x}^2 \right)^2 + F_0\hat{x}, \quad (3)$$

where  $\delta = 1 - \nu/\Omega$ ,  $\tilde{m} = m/\delta$ , and  $\tilde{\Omega} = \Omega\delta$ . In phase space, the extremal points of  $\hat{H}_s^{(\delta)}$  satisfies

$$p = 0, \quad x^3 - \frac{2\tilde{m}\tilde{\Omega}^2}{3\gamma}x - \frac{2F_0}{3\gamma} = 0. \quad (4)$$

For this cubic equation, the discriminant reads  $\Delta = [-2F_0/(3\gamma \times 2)]^2 + [-2\tilde{m}\tilde{\Omega}^2/(3\gamma \times 3)]^3$ . If and only if  $\Delta < 0$ , there exist three different real roots. This implies

$$F_0 < \frac{2}{9} \left( \frac{2\tilde{m}^3\tilde{\Omega}^6}{\gamma} \right)^{1/2} = \frac{8\sqrt{3}m\Omega^2\delta^{3/2}}{9} = F_c. \quad (5)$$

This result, based on the existence condition of multiple steady states, coincides with that from the singularity analysis [3]. The advantage of the present method is its applicability to more general situation, e.g., the mesoscopic case under present study, in which we will see that the *singular bifurcation* is absent. Nevertheless, in this work we still refer to the estimated  $F_c$  of Eq. (5) as *bifurcation point* for description convenience, particularly as a reference value for the driving strength.

Moreover, still classically, to realize the bifurcation it was found in Ref. [3] that the driving frequency  $\nu$  should be lower than  $\Omega$  and satisfy  $\delta > \sqrt{3}/(2Q)$ , where  $Q = \Omega/\kappa$ . Based on the above analysis, here we can further set a upper limit to the detuning. For  $F_0 < F_c$ , Eq. (4) has three different real roots, with the largest one as  $X_{\max} = 2[F_c/(3\gamma)]^{1/3} \cos \theta/3$ , where

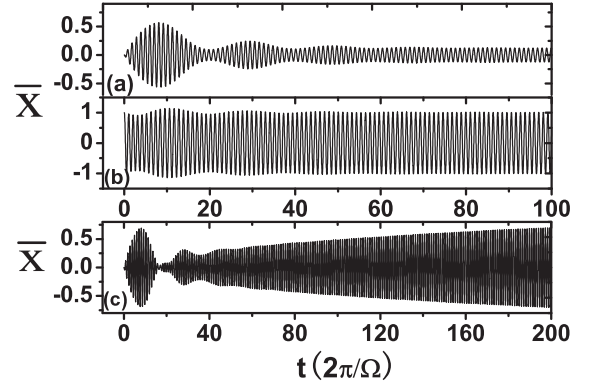


FIG. 1: (a) and (b): Bistable feature visualized by the transient behavior of  $\bar{x}(t) = \text{Tr}[\hat{x}\rho(t)]$ . Parameters: driving strength  $F_0 = 0.8F_c$ , and driving frequency  $\nu = 0.94\Omega$ . (c): Two successive stages (i.e. quenching and escape) towards the LAS, for driving near the critical point as exemplified here by  $F_0 = 0.95F_c$ .

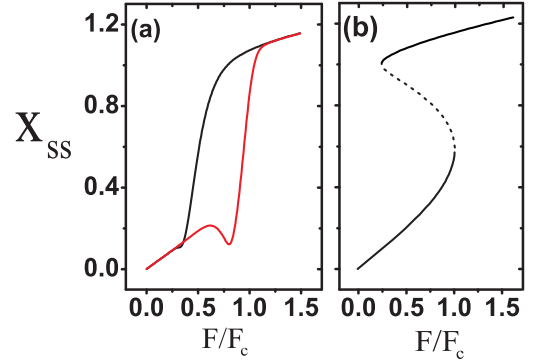


FIG. 2: (a) Phase diagram of the oscillation amplitude of stable state against the driving strength. The red and black curves are for initial states of Gaussian wavepackets centered at  $x = 0$  and  $x = 1$ , respectively. (b) Classical counterpart of (a), showing sharp bifurcation behavior.

$\theta = \arctan \sqrt{(F_c/F_0)^2 - 1}$ . Obviously, this largest amplitude should not overcome the potential barrier of the Duffing oscillator. This consideration leads to the following inequality:

$$2\left(\frac{F_c}{3\gamma}\right)^{1/3} < \sqrt{m\Omega^2/(4\gamma)} \implies \delta < \frac{9}{64(3\tilde{\gamma})^{1/3}}, \quad \tilde{\gamma} = \frac{\gamma}{m\Omega^2}.$$

In our model,  $\gamma = m\Omega^2/24$ . So we get  $\delta < 18/64 \approx 0.28$ . This is not the optimized value. More accurate consideration can result in even smaller upper limit.

*Quantum Dynamics near the Bifurcation Point.*— Based on a direct simulation of the master equation, we show in Fig. 1 the evolution of  $\bar{x}(t) = \text{Tr}[\hat{x}\rho(t)]$ . First, in Fig. 1(a) and (b), we demonstrate the *bistable* nature, for driving strength  $F_0 = 0.8F_c$  as an example. We consider two initial conditions: the ground state, and a coherent state centered at  $\bar{x} = 1$ . Indeed, for the mesoscopic DDO, here we find *quantum mechanically* that the steady state does exhibit bistable behavior, say, depending on the initial condition, it arrives at either a small amplitude state (SAS), or a large amplitude state

(LAS). Nevertheless, as we will understand later in the following study, the results in Fig. 1(a) and (b) are not the *fundamental* SAS and LAS, but their mixture with different population probabilities depending on the initial conditions. For driving not very close to  $F_c$ , the “steady state” population is formed in relatively short time, and later the fluctuation-induced transitions between the fundamental SAS and LAS are negligibly weak.

In contrast, as shown in Fig. 1(c), for driving closer to  $F_c$  (e.g.  $F_0 = 0.95F_c$ ) we find that the entire process contains two distinct stages, say, a (fast) *quenching* stage, and a successive (slow) *escape* stage. In the quenching stage, the oscillator rapidly arrives at the SAS. Then, it is followed by a rate process (transition) to the LAS, which may be termed as the Kramers *escape* process [12]. Conventionally the escape is caused by thermal fluctuations, as described by for instance the mean-first-passage-time approach or Fokker-Planck equation [8, 9]. In Ref. [9], quantum-fluctuation induced transition was also investigated, by using the WKB method. The advantage of our present numerical simulation allows to formulate a way to extract the transition rate under more general conditions, say, in the presence of both thermal and quantum fluctuations, going beyond the existing results in limiting cases. This will be detailed in latter analysis.

In Fig. 2(a) we extract data from numerical simulation as shown in Fig. 1 to plot the *phase diagram*, say, the oscillation amplitude of steady state against the driving strength, which shows the desirable *hysteresis* behavior. However, compared to its classical counterpart as schematically shown in Fig. 2(b), two differences should be remarked. (i) The singularity of transition from the bistable region to the single LAS or SAS disappears, although the transition point, i.e.,  $F_c = (8\sqrt{3}m\Omega^2\delta^{3/2})/9$ , is approximately preserved. The basic reason for this *gradual* transition behavior is that for the present *mesoscopic* DDO, the stable state is a statistical mixture of the SAS and LAS in the absence of noise (i.e. thermal and quantum fluctuations). (ii) A dip appears in the red curve. Our numerical simulation shows that the time-dependent oscillations of the SAS and LAS are out of phase (i.e. with a phase difference about  $\pi$ ), and the SAS and LAS themselves depend on the driving strength *nonlinearly*. As a consequence of interplay of these two factors, the *dip* is formed as we observed.

Let us proceed our further analysis with the help of the Wigner function, which is defined as:  $W(x, p, t) = 1/(\pi\hbar) \int_{-\infty}^{+\infty} \langle x + x' | \rho(t) | x - x' \rangle \exp(-i2px'/\hbar) dx'$ . The Wigner function is widely used in broad context of physics, with an intuitive interpretation of probability in the phase space. In Fig. 3(a) we show the Wigner function of the oscillator at time  $t = 160 * (2\pi/\Omega)$ , for driving strength  $F_0 = 0.9F_c$ , and with the ground state as the initial condition. Time dependently, the Wigner function is in fact rotating with the driving frequency in phase space, along the classical trajectory but with additional diffusion because of the thermal and/or quantum fluctuations.

In the transient process, after certain duration time to be dis-

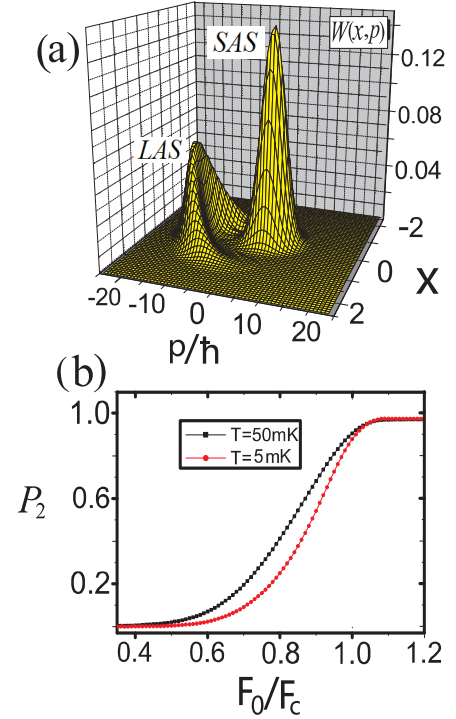


FIG. 3: (a) Wigner function at  $t = 160 * (2\pi/\Omega)$ , for driving  $F_0 = 0.9F_c$  and starting with the ground state. The two separated wavepackets correspond to the SAS and LAS, and the SAS is nearly a perfect coherent state. (b) Occupation probability of the LAS ( $P_2$ ) against the driving strength. In the region near  $F_c$ ,  $P_2$  shows exponential dependence of the driving strength.

cussed below, the oscillator can be well described by a mixed state, formally as

$$W(x, p, t) = P_1(t)W_S(x, p, t) + P_2(t)W_L(x, p, t). \quad (6)$$

Here,  $W_S(x, p, t)$  and  $W_L(x, p, t)$  are, respectively, the Wigner functions of the *intrinsic* SAS and LAS associated with the given driving strength, but not the *averaged* ones shown in Fig. 2(a). In essence, the SAS and LAS are two limit cycles, or attractors, each with its own attraction basin [1, 13].  $P_1(t)$  and  $P_2(t)$  are the occupation probabilities of the SAS and LAS. In Fig. 3(b) we plot  $P_2(t)$  at  $t = 160 * (2\pi/\Omega)$  versus the driving strength, for two temperatures.

Closer inspection indicates that  $W_S(x, p, t)$  is quantum mechanically a pure state, i.e.,  $\text{Tr} \rho_{SAS}^2 \simeq 1$ , where  $\rho_{SAS}$  is the density matrix of the SAS. Moreover, it is nearly a perfect coherent state. This result can be understood as follows. For a harmonic oscillator under the interplay of driving and dissipation, such as the optical cavity field under excitation and photon-loss, the steady state is exactly a coherent state [14]. Then, for the present *nonlinear* Duffing oscillator, since the SAS is not far from the oscillator origin, it is thus approximately governed by a harmonic oscillation. For LAS, however, which is far from the origin, nonlinearity is prominent, which makes  $W_L(x, p, t)$  not at all a coherent state, but a mixed state with partial coherence.

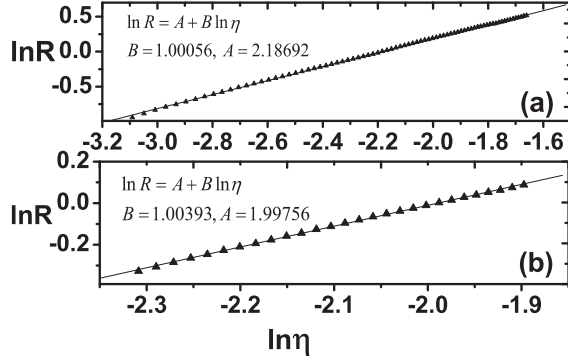


FIG. 4: Perfect scaling behavior hidden in the transition rate with the driving distance to the critical point, say,  $\eta = F_c^2 - F_0^2$ . The triangles stand for data from numerical simulation, and the straight lines are linear fits. The result shows that  $R \propto \eta^\alpha$ , and  $\alpha \simeq 1$ . Parameters:  $\kappa = 0.01$ ;  $T=5\text{mK}$  in (a), and  $T=50\text{mK}$  in (b).

*Transition Rate.*— Now we return to the transient dynamics and focus on the *escape stage* as indicated in Fig. 1(c). This stage is described by a rate process:

$$\frac{dP_1(t)}{dt} = -\kappa_1 P_1 + \kappa_2 P_2, \quad \frac{dP_2(t)}{dt} = -\kappa_2 P_2 + \kappa_1 P_1, \quad (7)$$

where  $\kappa_1$  is the escape rate from SAS to LAS, and  $\kappa_2$  vice versa. In what follows, we formulate a way to determine the escape rate, by contacting Eq. (7) with the numerical simulation. First, the solution of Eq. (7) reads

$$P_1(t) = \frac{\kappa_2}{\kappa_1 + \kappa_2} + \left[ P_1(0) - \frac{\kappa_2}{\kappa_1 + \kappa_2} \right] \exp^{-(\kappa_1 + \kappa_2)t}, \quad (8)$$

$$P_2(t) = \frac{\kappa_1}{\kappa_1 + \kappa_2} + \left[ P_2(0) - \frac{\kappa_1}{\kappa_1 + \kappa_2} \right] \exp^{-(\kappa_1 + \kappa_2)t}. \quad (9)$$

$P_1(0)$  and  $P_2(0)$  are certain “initial” values of the population probabilities in the escape stage. Rather than  $P_1(0)$  and  $P_2(0)$ , which are not well defined, we use the probabilities  $P_2(t_j)$  at three time points, and simply assume  $t_3 - t_2 = t_2 - t_1 = \Delta t$ . Then, based on Eqs. (8) and (9), we obtain

$$\kappa_1 = -\frac{[P_2(t_3) - (K-1)P_2(t_1)] \ln(K-1)}{[1 - (K-1)^2] \Delta t}, \quad (10)$$

$$\kappa_2 = -\frac{\ln(K-1)}{\Delta t} - \kappa_1, \quad (11)$$

where  $K = [P_2(t_3) - P_2(t_1)]/[P_2(t_2) - P_2(t_1)]$ .

Below we focus on  $\kappa_1$  and formally assume  $\kappa_1 = C e^{-R/\lambda}$ . Here  $C$  is an irrelevant prefactor, and the exponential form of  $\sim e^{-R/\lambda}$  is associated with an *effective* activation process. In limiting cases, such as for classical thermal activation,  $R$  is the activation energy and  $\lambda$  the temperature; while for quantum tunneling through a barrier,  $R$  is the tunneling action and  $\lambda$  the Planck constant. Our present situation is a generalization, i.e., quantum-dynamical-tunneling dominated but also thermal-activation involved. So, we may view  $R$  as an effective activation energy and  $\lambda$  an effective Planck constant or temperature.

Physically, we should expect that the transition rate depends on the driving distance to the critical point  $F_c$ , i.e.,

$\eta \equiv F_c^2 - F_0^2$ , since closer to the critical point, more easily can the transition to the LAS take place. Strikingly, Figure 4 displays a perfect scaling behavior for this dependence. Assuming  $R \propto \eta^\alpha$ , our precise numerical fitting gives  $\alpha \simeq 1$ . We noticed similar scaling behavior was found by Dykman [8], but where a scaling exponent  $\alpha = 3/2$  was found instead.

Since our simulation is for a mesoscopic DDO with a bit more than ten levels involved in the dynamics, we postulate that the scaling exponent  $\alpha = 3/2$  is not universal. As in [5] and [10], our present simulation does not account for the driving field in the dissipation terms. Although this kind of treatment is well accepted in vast areas, there is indeed some counterexamples, for instance, see the most recent Ref. [15]. Nevertheless, by transforming the system to a rotating frame to account for the driving in the dissipation terms and calculating the fidelity of the SAS, we actually discovered the same scaling exponent [16].

We noticed that in Ref. [17], scaling behavior of the transition rate with the driving frequency (but not the driving strength) was analyzed to give  $\alpha \simeq 1.3 \sim 1.4$ , by a rough fitting from a few experimental data. Meanwhile, in the experiments by Siddiqi *et al.* [18], an effective potential with a barrier height scaled as  $\Delta U_{dyn}^0 \propto [1 - (F_0/F_c)^2]^{3/2}$ , was employed to analyze their measured data by means of the *thermal-activation* rate  $\propto \exp(-\Delta U_{dyn}^0/k_B T)$ . Based on the same effective potential, a rough WKB analysis should result in a smaller scaling exponent for *quantum* rate  $\propto \exp(-\sqrt{\Delta U_{dyn}^0} a/\hbar)$ , with  $a$  the effective width of the barrier. It seems that our above result  $\alpha \simeq 1$  is in qualitative agreement with this analysis. Therefore, stronger experimental evidences and more rigorous theoretical investigations will be helpful to clarify this interesting issue, particularly with an extension to the mesoscopic regime as we estimated earlier in this work.

*Summary and Discussion.*— We have investigated the quantum nonlinear dynamics of the DDO in a mesoscopic regime with a few more than ten energy levels involved in the dynamics. We demonstrated for the first time that the quantum nature significantly modifies the classical sharp-switching behavior near the bifurcation point. In terms of Wigner function the state near the bifurcation point was identified to be a mixture of low- and high-amplitude states, and the low-amplitude component is nearly a perfect coherent state. More interestingly, near the bifurcation point the transient dynamics reveals distinct stages of *quenching* and *escape*. The latter is well characterized by a rate process, and the numerically extracted rate displays perfect scaling behavior with the driving distance to the bifurcation point.

The quantum predictions of this work, in particular the scaling exponent that differs from the existing result [8], raise an open question and deserve further investigations. The statistically mixed nature of the state near the bifurcation point may also partially explain the discrepancy between the classical prediction and the measurement data in the JBA experiment [2]. Related to this, the mesoscopic DDO may also support quantum weak measurement in the transient stage, where

qubit state can be updated using a generalized Bayesian rule.

*Acknowledgments.*— We acknowledge insightful discussions with Professor Gang Hu. This work was supported by the National Natural Science Foundation of China under grants No. 10874176, 10875011 and 10575010, the Major State Basic Research Project under No. 2006CB921201, the 973 Program under No.2007CB814805, and the Foundation of Doctoral Training under No. 20060027009.

- 
- [1] I. Siddiqi *et al.* Phys. Rev. Lett. **84**, 207002 (2004); I. Siddiqi *et al.* Phys. Rev. Lett. **94**, 027005 (2005).
  - [2] I. Siddiqi *et al.* Phys. Rev. B **73**, 054510 (2006); N. Boulant *et al.* Phys. Rev. B **76**, 014525 (2007).
  - [3] V. E. Manucharyan *et al.* Phys. Rev. B **76**, 014524 (2007).
  - [4] A. Lupascu *et al.* Nat. Phys. **3**, 119 (2007).
  - [5] I. Katz, A. Retzker, R. Straub, and R. Lifshitz Phys. Rev. Lett. **99**, 040404 (2007).
  - [6] V. Peano and M. Thorwart Chem. Phys. **322**, 135 (2006).
  - [7] M. Marthaler and M. I. Dykman Phys. Rev. A **73**, 042108 (2006).
  - [8] M. I. Dykman Phys. Rev. E **75**, 011101 (2007).
  - [9] I. Serban and F. K. Wilhelm Phys. Rev. Lett. **99**, 137001 (2007).
  - [10] M. A. Armen and H. Mabuchi Phys. Rev. A **73**, 063801 (2006).
  - [11] YiJing Yan Phys. Rev. A **58**, 2721 (1998).
  - [12] P. Hanggi, P. Talkner, and M. Borkovec Rev. Mod. Phys. **62**, 251 (1990).
  - [13] I. Kozinsky *et al.* Phys. Rev. Lett. **99**, 207201 (2007).
  - [14] C. Gerry and P. Knight, *Introductory Quantum Optics*, P. 208, (Cambridge University Press, 2005)
  - [15] D.W. Hone, R. Ketzmerick, and W. Kohn Phys. Rev. E **79**, 051129 (2009).
  - [16] L.Z. Guo, Z.G. Zheng and X.Q. Li (unpublished).
  - [17] C. Stambaugh and H.B. Chan Phys. Rev. B **73**, 172302 (2006).
  - [18] I. Siddiqi, R. Vijay, F. Pierre, C.M. Wilson, M. Metcalfe, C. Rigetti, L. Frunzio, and M.H. Devoret arXiv:cond-mat/0312623; I. Siddiqi, R. Vijay, F. Pierre, C.M. Wilson, L. Frunzio, M. Metcalfe, C. Rigetti, and M.H. Devoret arXiv:cond-mat/0507248.